Conductors and Boundary Conditions

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Agenda

- Intended learning outcomes.
- Current and current density
- Continuity of current
- Quantum Theory
- Conductivity and superconductors
- Ohm’s law
- Conductor properties and boundary conditions
- Streamlines and Sketches of Fields
- Electric dipole
- The method of images

Intended learning outcomes

- **I**n this chapter, we apply the methods we have learned to some of the materials with which an engineer must work.
- In the first part of the chapter, we consider conducting materials by describing the parameters that relate current to an applied electric field. This leads to a general definition of Ohm's law.
- We then develop methods of evaluating resistances of conductors in a few simple geometric forms. Conditions that must be met at a conducting boundary are obtained next, and this knowledge leads to a discussion of the method of images.
- The properties of semiconductors are described to conclude the discussion of conducting media.

Current and Current Density

Electric charges in motion constitute a *current*.

The unit of current is the ampere (A),

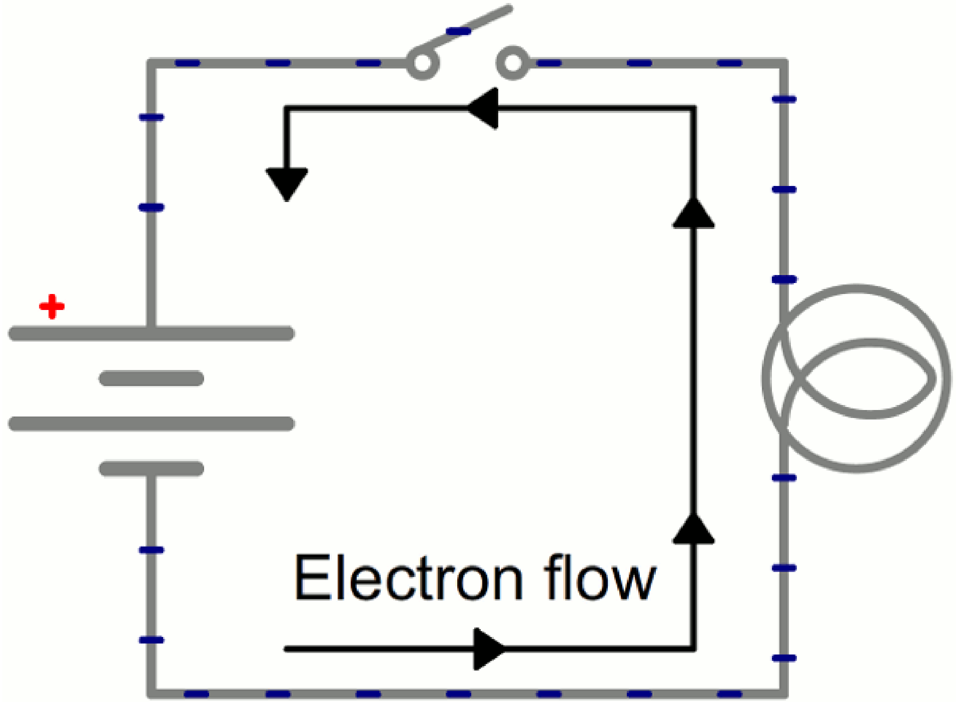
It is defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second.

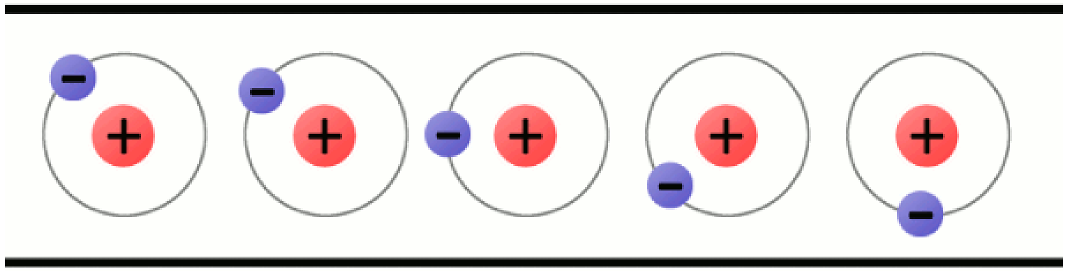
Current is symbolized by I, and given by

$$I = \frac{dQ}{dt} \qquad (1)$$

- Current is usually defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons.

Current and Current Density





Current flow inside conductor

Current and Current Density

In field theory, we are usually interested in events occurring at a point rather than within a large region, and we find the concept of *current density,* measured in amperes per square meter (A/m^2), more useful. Current density is a vector represented by **J**.

The increment of current I crossing an incremental surface S normal to the current density is

$$\Delta I = J_N \Delta S$$

and in the case where the current density is not perpendicular to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \qquad (2)$$

Current and Current Density

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$, as shown in Figure 5.1$a$.

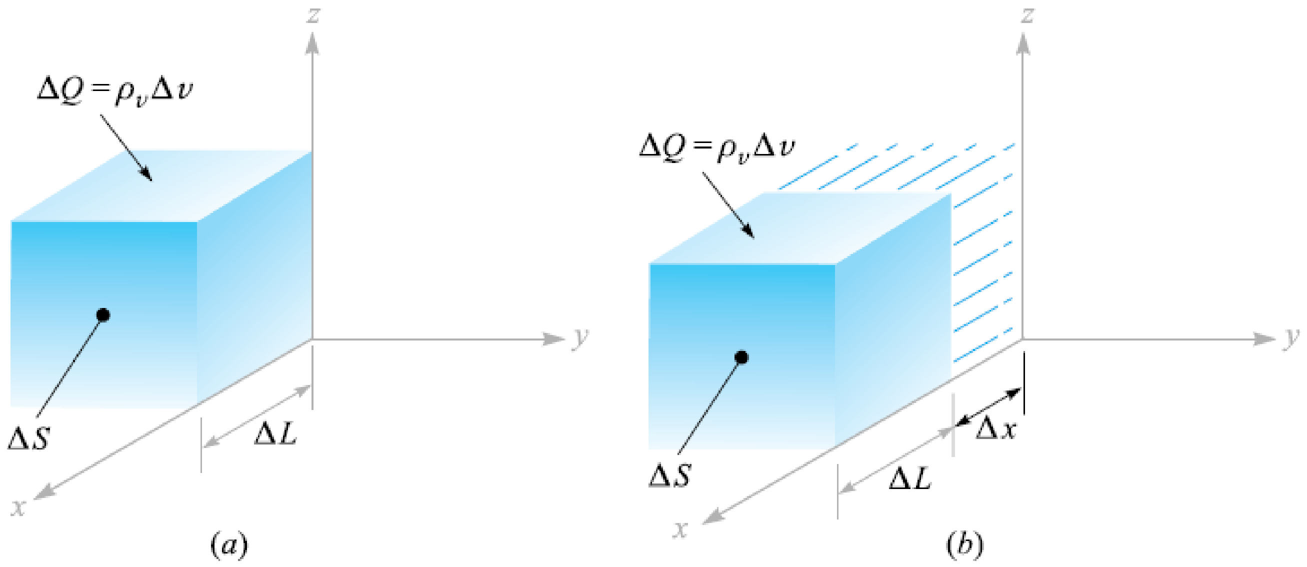


Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

Current and Current Density

To simplify the explanation, assume that the charge element is oriented to the x-axis and has only an x component of velocity $\Delta Q = \rho_v \Delta S \Delta x$.

If the charge element ΔQ moved a distance Δx in the time interval Δt, as indicated in Figure 5.1b, the resulting current will be

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

$$\Delta I = \rho_v \, \Delta S \, v_x$$

Where v_x represents the x component of the velocity **v**. In terms of current density, we find

$$J_x = \rho_v \, v_x$$

and in general

$$\mathbf{J} = \rho_v \mathbf{v} \qquad (3)$$

Current and Current Density

This last result shows clearly that charge in motion constitutes a current. We call this type of current a *convection current.*

Note that the convection current density is related linearly to charge density as well as to velocity.

The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

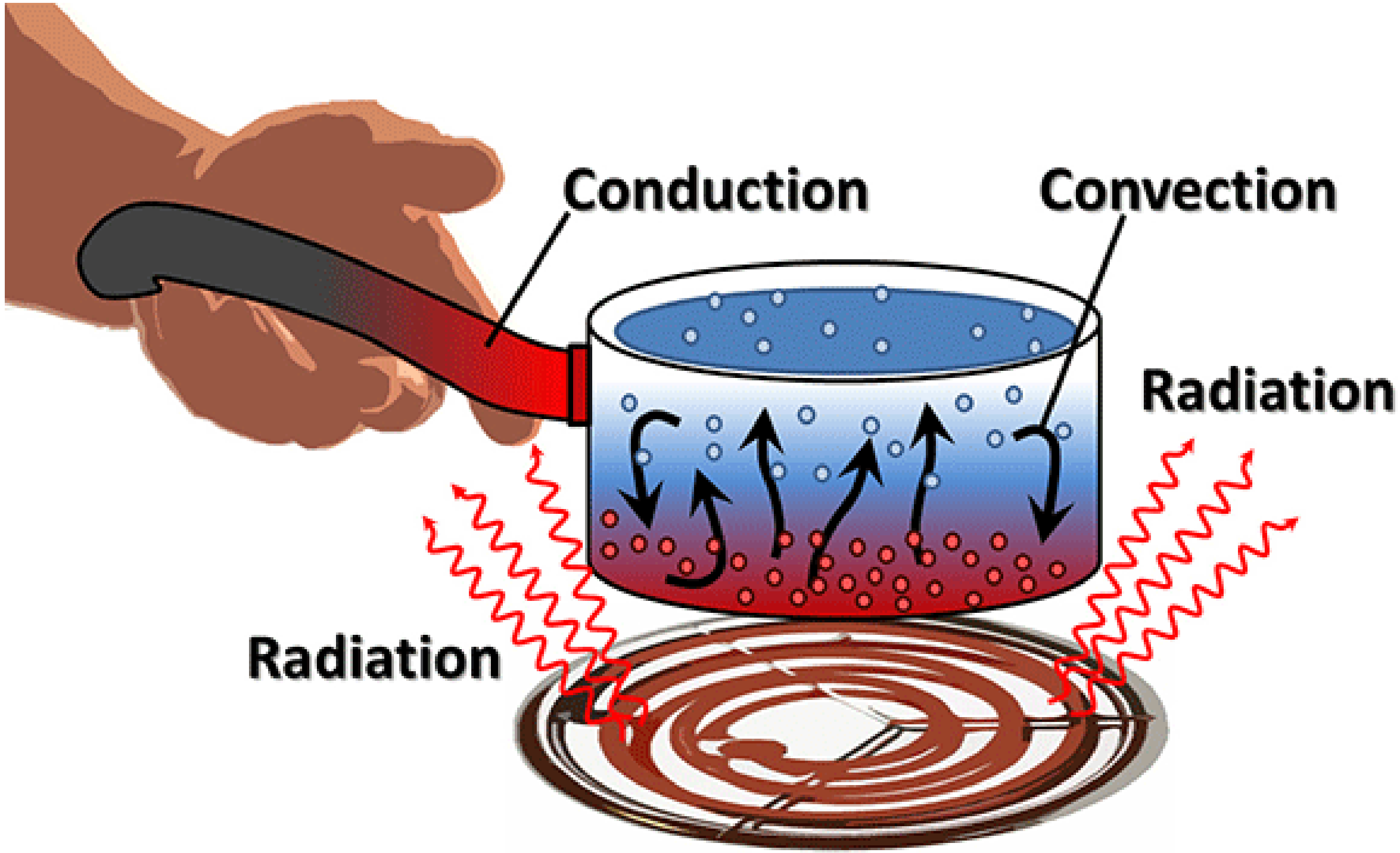
Current and Current Density

There are 2 types of current:

1) Convection current
   - generated by actual movement of electrically charged matter; does NOT obey Ohm’s law
   - E.g. movement of charged particles in cathode ray tube

2) Conduction current
   - atoms of conducting material do NOT move; obeys Ohm’s law
   - E.g. movement of electrons in a metal wire

Current and Current Density



Current and Current Density

D5.1. Given the vector current density $\mathbf{J} = 10\rho^2 z\mathbf{a}_\rho - 4\rho\cos^2\phi\,\mathbf{a}_\phi$ mA/m^2: (*a*) find the current density at $P(\rho = 3, \phi = 30^\circ, z = 2)$; (*b*) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180\mathbf{a}_\rho - 9\mathbf{a}_\phi$ mA/m^2; 3.26 A

Continuity of Current

- the continuity equation of current is introduced from the concept of "conservation of charge".
- The principle of conservation of charge states simply that "charges can be neither created nor destroyed", although equal amounts of positive and negative charge may be *simultaneously* created, obtained by separation, or lost by recombination.
- when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

- This *outward flow* of positive charge must be balanced by a decrease of positive charge within the closed surface and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \qquad (4)$$

If the charge inside the closed surface is Q_i, then the rate of decrease is $-dQ_i/dt$

Continuity of Current

Equation (4) is the integral form of the continuity equation; the differential, or point form, is obtained by using the divergence theorem:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v \, dv$$

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t} \qquad (5)$$

Continuity of Current

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

D5.2. Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5}\mathbf{a}_z$ A/m^2 in the region $0 \leq \rho \leq 20\ \mu$m; for $\rho \geq 20\ \mu$m, $\mathbf{J} = 0$. (*a*) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction. (*b*) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_v there. (*c*) If the volume charge density at $z = 0.15$ m is -2000 C/m^3, find the charge velocity there.

Ans. $-39.7\ \mu$A; -15.8 mC/m^3; 29.0 m/s

Continuity of Current

Ex 1: Assume that an electron beam carries a total current of -500 μA in the $\bar{a}_z$direction and has a current density $J_z$that is not a function of r or $\emptyset$ in the region $0 \leq r < 10^{-4} m$ and zero for $r > 10^{-4} m$.

If the electron velocity is given by $V_z = 8 \times 10^7 z \frac{m}{s}$, calculate ρ_v at $r = 0$ and (a) z = 1 mm; (b) z = 2 cm; (c) z = 1 m.

Sol.

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$ and $$\mathbf{J} = \rho_v \mathbf{v}$$

The above relations will be used to calculate ρ_v where v and I are known.
Given that $J = 0$ for $r > 10^{-4} m$, and the current density is directed parallel to the z-direction. Therefore, the cross-section area for the current is $S_{xy} = \pi(10^{-4})^2$ m^2.

$$J_z = \frac{I}{S_{xy}} = -\frac{500 \times 10^{-6}}{\pi \times 10^{-8}} \text{ A/m}^2$$

Continuity of Current

$J_z = \rho_v V_z$, V_z= 8x10^{7}z

$$\therefore -\frac{500 \times 10^{-6}}{\pi \times 10^{-8}} = \rho_v \times 8 \times 10^7 z$$

$$\therefore \rho_v = -\frac{0.005}{8\pi z} \text{ c/m}^3$$

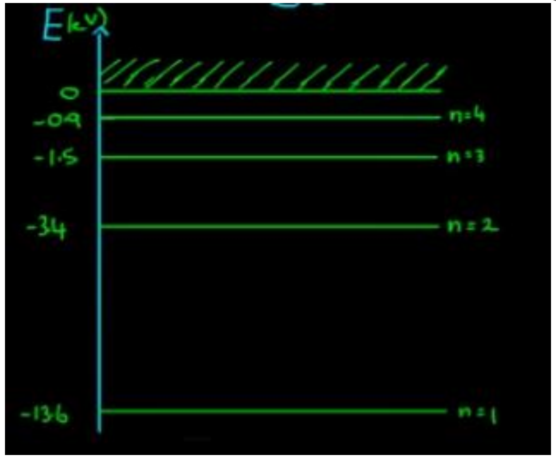
For (a) z = 1 mm; $\therefore \rho_v$ = -0.1989 c/m^3

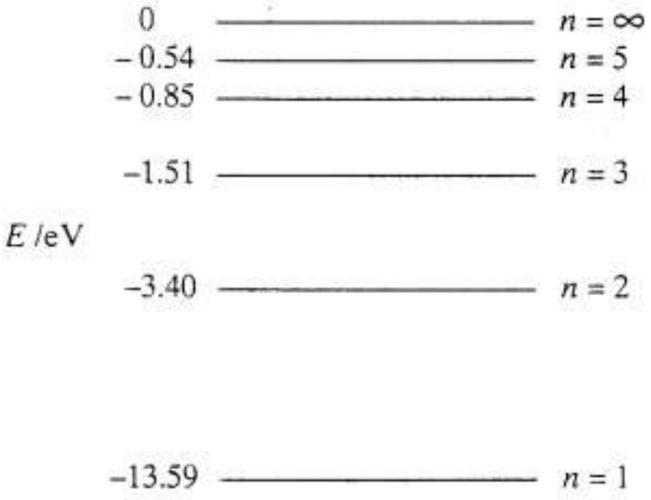
For (b) z = 2 cm; $\therefore \rho_v$ = -9.95x10^{-3} c/m^3

For (c) z = 1 m. $\therefore \rho_v$ = -0.1989x10^{-3} c/m^3

Quantum Theory

Physicists describe the behavior of the electrons surrounding the positive atomic nucleus in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance from the nucleus. The total energy is the sum of the kinetic and potential energies, and because energy must be given to an electron to pull it away from the nucleus, the energy of every electron in the atom is a negative quantity.





Quantum Theory

According to the quantum theory, only certain discrete energy levels, or energy states, are permissible in a given atom, and an electron must therefore absorb or emit discrete amounts of energy, or quanta, in passing from one level to another.

In a crystalline solid, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the allowed energies of electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels.

At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the *valence band*.

Quantum Theory

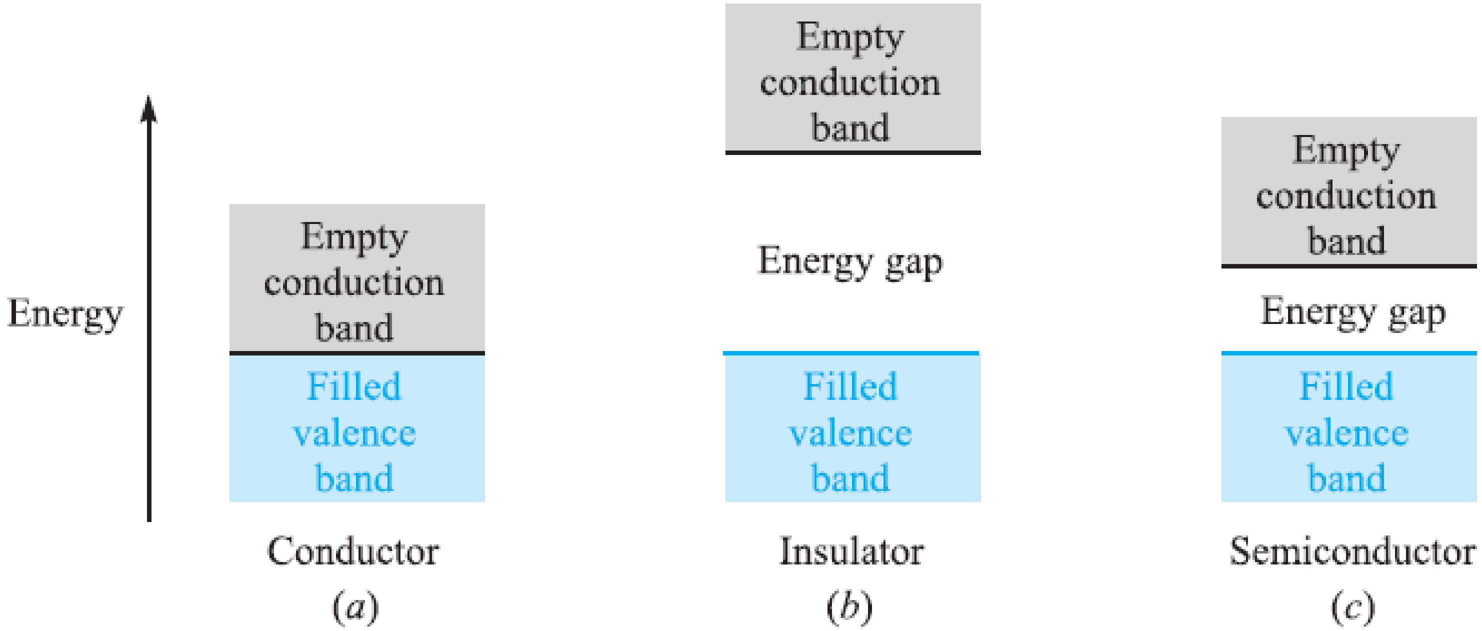


Figure 5.2 The energy-band structure in three different types of materials at 0 K. (*a*) The conductor exhibits no energy gap between the valence and conduction bands. (*b*) The insulator shows a large energy gap. (*c*) The semiconductor has only a small energy gap.

Quantum Theory

In **good conductors,** if an external field is applied, additional kinetic energy are given to the valence electrons, then the valence band merges smoothly into a *conduction band*, resulting in an electron flow.

In **good insulators,** because there is a big energy gab, the electron cannot accept additional energy in small amounts. This band structure is indicated in Figure 5.2b. Note that if a relatively large amount of energy can be transferred to the electron, it may be sufficiently excited to jump the gap into the next band where conduction can occur easily. Here the insulator breaks down.

In **semiconductors,** only a small energy gab (forbidden region) separates the two bands, as illustrated by Figure 5.2c. Small amounts of energy in the form of heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called semiconductors.

Conductivity and superconductors

If a free electron, (Q = −e), moves under the influence of an electric field **E**, the electron will experience a force **F**

$$\overline{F} = \mathrm{Q}\,\overline{E} = -\mathrm{e}\,\overline{E}$$

In free space, the electron would accelerate and continuously increase its velocity (and energy); In the crystalline material, the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity $\mathbf{v}_d$ is termed the *drift velocity,*

$$\overline{\mathrm{v}}_d = -\mu_e \overline{E} \qquad (6)$$

where μ_e is the mobility of an electron and is positive by definition. Note that the electron velocity is in a direction opposite to the direction of **E**. the unit of mobility is m^2/V.s

Conductivity and superconductors

Substituting (6) into Eq. (3), we obtain

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E} \qquad (7)$$

where ρ_e is the free-electron charge density, a negative value. The total charge density ρ_v is zero because equal positive and negative charges are present in the neutral material.

The relationship between **J** and **E** for a metallic conductor, however, is also specified by the conductivity σ (sigma),

$$\mathbf{J} = \sigma \mathbf{E} \qquad (8)$$

We call this equation the *point form of Ohm's law.* The unit of σ is Siemens per meter (S/m) or mho per meter (℧/m)

Conductivity and superconductors

If we now combine Equations (7) and (8), conductivity may be expressed in terms of the charge density and the electron mobility as

$$\sigma = -\rho_e \mu_e \qquad (9)$$

From the definition of mobility (6), it is now satisfying to note that a higher temperature infers a greater crystalline lattice vibration, more impeded electron progress for a given electric field strength, lower drift velocity, lower mobility, lower conductivity from Eq. (9), and higher resistivity as stated.

Conductivity and superconductors

Superconductivity

The resistivity, which is the reciprocal of the conductivity, varies almost linearly with temperature in the region of room temperature, and for aluminum, copper, and silver it increases about 0.4 percent for a 1-K rise in temperature. For several metals the resistivity drops abruptly to zero at a temperature of a few kelvin; this property is termed superconductivity.

Copper and silver are not superconductors, although aluminum is (for temperatures below 1.14 K).

Ohm's law

The application of Ohm's law in point form to a macroscopic (visible to the naked eye) region leads to a more familiar form. Initially, assume that **J** and **E** are *uniform*, as they are in the cylindrical region shown in Figure 5.3.

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS \qquad (10)$$

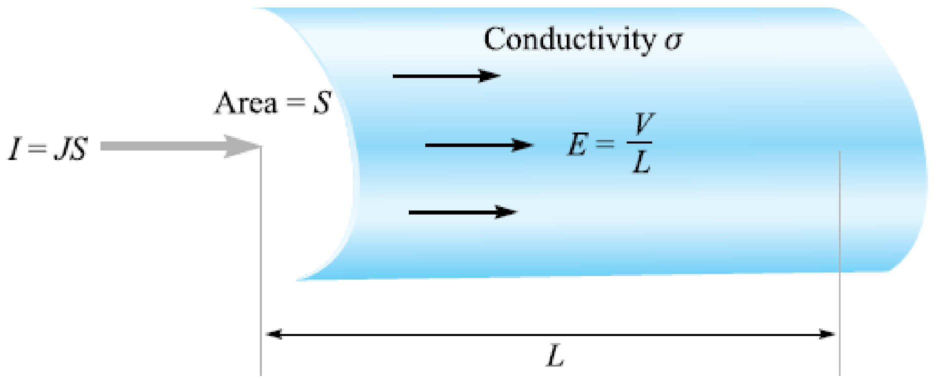


Figure 5.3 Uniform current density J and electric field intensity E in a cylindrical region of length L and cross-sectional area S. Here $V = IR$, where $R = L/\sigma S$.

Ohm's law

Because they are uniform, and

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba} \qquad (11)$$

$$= \mathbf{E} \cdot \mathbf{L}_{ab}$$

or

$$V = EL$$

Thus

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

$$V = \frac{L}{\sigma S} I$$

Ohm's law

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized from elementary circuit theory as the *resistance* of the cylinder, and therefore

$$V = IR \qquad (12)$$

$$R = \frac{L}{\sigma S} \qquad (13)$$

- Equation (12) is known as *Ohm's law*, and Eq. (13) enables us to compute the resistance *R*, measured in ohms (Ω), of conductors which possess uniform fields.
- when the fields are non-uniform

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}} \qquad (14)$$

- The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equipotentials.

Ohm's law

As an example of the determination of the resistance of a cylinder, we find the resistance of a 1-mile length of #16 copper wire, which has a diameter of 0.0508 in.

Solution. The diameter of the wire is $0.0508 \times 0.0254 = 1.291 \times 10^{-3}$ m, the area of the cross section is $\pi(1.291 \times 10^{-3}/2)^2 = 1.308 \times 10^{-6}$ m^2, and the length is 1609 m. Using a conductivity of 5.80×10^7 S/m, the resistance of the wire is, therefore,

$$R = \frac{1609}{(5.80 \times 10^7)(1.308 \times 10^{-6})} = 21.2\ \Omega$$

This wire can safely carry about 10 A dc, corresponding to a current density of $10/(1.308 \times 10^{-6}) = 7.65 \times 10^6$ A/m^2, or 7.65 A/mm^2. With this current, the potential difference between the two ends of the wire is 212 V, the electric field intensity is 0.312 V/m, the drift velocity is 0.000 422 m/s, or a little more than one furlong a week, and the free-electron charge density is -1.81×10^{10} C/m^3, or about one electron within a cube two angstroms on a side.

Ohm's law

D5.3. Find the magnitude of the current density in a sample of silver for which $\sigma = 6.17 \times 10^7$ S/m and $\mu_e = 0.0056\ \text{m}^2/\text{V}\cdot\text{s}$ if (*a*) the drift velocity is 1.5 μm/s ; (*b*) the electric field intensity is 1 mV/m; (*c*) the sample is a cube 2.5 mm on a side having a voltage of 0.4 mV between opposite faces; (*d*) the sample is a cube 2.5 mm on a side carrying a total current of 0.5 A.

Ans. 16.5 kA/m^2; 61.7 kA/m^2; 9.9 MA/m^2; 80.0 kA/m^2

D5.4. A copper conductor has a diameter of 0.6 in. and it is 1200 ft long. Assume that it carries a total dc current of 50 A. (*a*) Find the total resistance of the conductor. (*b*) What current density exists in it? (*c*) What is the dc voltage between the conductor ends? (*d*) How much power is dissipated in the wire?

Ans. 0.035 Ω; 2.74×10^5 A/m^2; 1.73 V; 86.4 W

Conductor Properties and Boundary Conditions

There are important properties for conductors:

1- ρ_v = 0, there is no charge inside the conductor.

2- E_{inside}= 0, there is no field inside the conductor.

3- E_{t}= 0, there is no tangential field component to the conductor surface.

4- surface charges may be exist.

5- $E_n = \rho_{\mathrm{s}}/\varepsilon$ may be exist to the conductor surface.

6- the conductor surface is an equipotential surface.

Conductor Properties and Boundary Conditions

Why ρ_v = 0 inside the conductor?

Suppose, for the sake of argument, that there suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away from each other. This continues until the electrons reach the surface of the conductor.

Hence the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface.

Conductor Properties and Boundary Conditions

Why $\boldsymbol{E_{inside}}$= 0 ?

Applying Gauss's law inside a conductor

$$\oint \overline{D}.\,d\overline{s} = Q_{enclosed} = \oint \rho_v dv$$

$\because \rho_v$ inside a conductor = 0

$$\therefore \oint \overline{D}.\,d\overline{s} = 0 \rightarrow \therefore D_{inside} = 0 \text{ and } E_{inside} = 0$$

Conductor Properties and Boundary Conditions

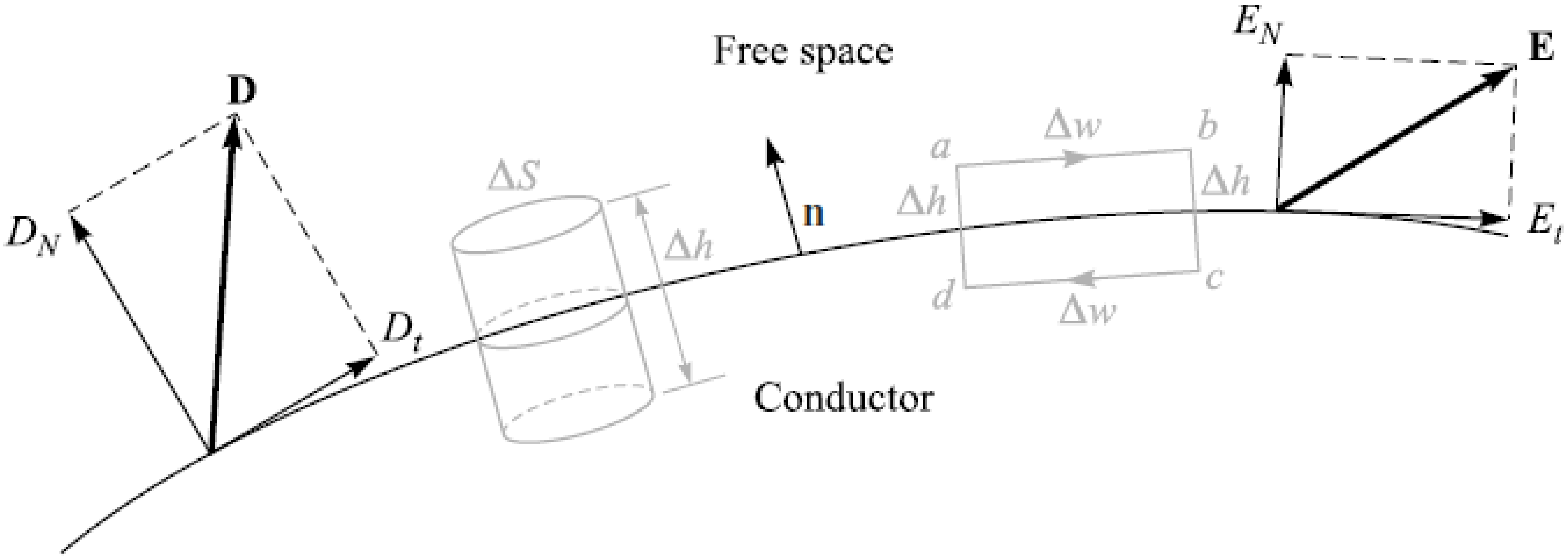
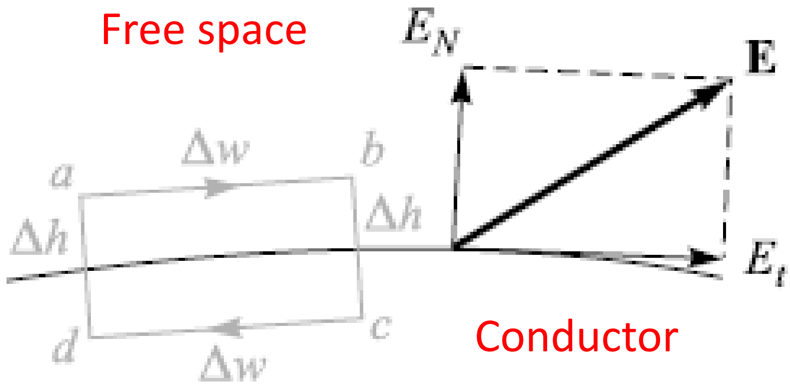


Figure 5.4 An appropriate closed path and gaussian surface are used to determine boundary conditions at a boundary between a conductor and free space; $E_t = 0$ and $D_N = \rho_S$.

Conductor Properties and Boundary Conditions

Why E_t= 0 ?



∵ E-field is conservative $\oint \mathbf{E} \cdot d\mathbf{L} = 0$

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

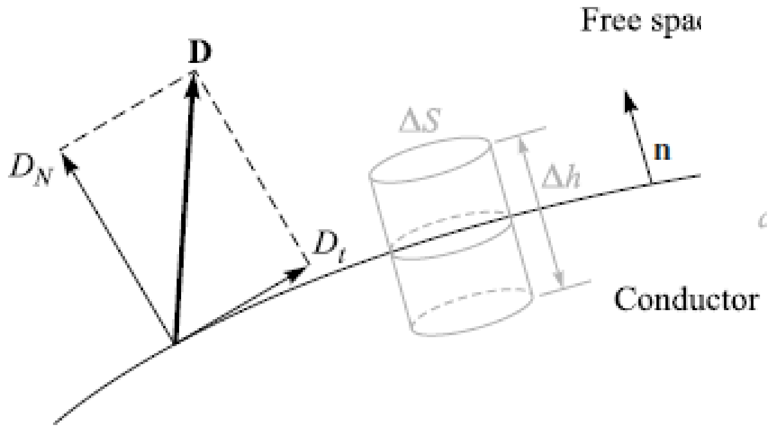
Remembering that E = 0 within the conductor,

$$E_t \Delta w - E_{N,\text{at } b} \frac{1}{2} \Delta h + E_{N,\text{at } a} \frac{1}{2} \Delta h = 0$$

$$\therefore E_t \Delta w = 0 \rightarrow \therefore E_t = 0$$

Conductor Properties and Boundary Conditions

Why $\boldsymbol{D_n = \rho_s}$



choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again, we let Δh approach zero.
Using Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

we integrate over the three distinct surfaces

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

Since E_{inside}= 0 and E_t= 0 → $\int_{\text{bottom}} = 0$ and $\int_{\text{sides}} = 0$

$$\therefore\ D_n\ \Delta S = Q_{enclosed} = \rho_S \Delta S$$

$$\boxed{\therefore\ D_n = \rho_S}$$

Conductor Properties and Boundary Conditions

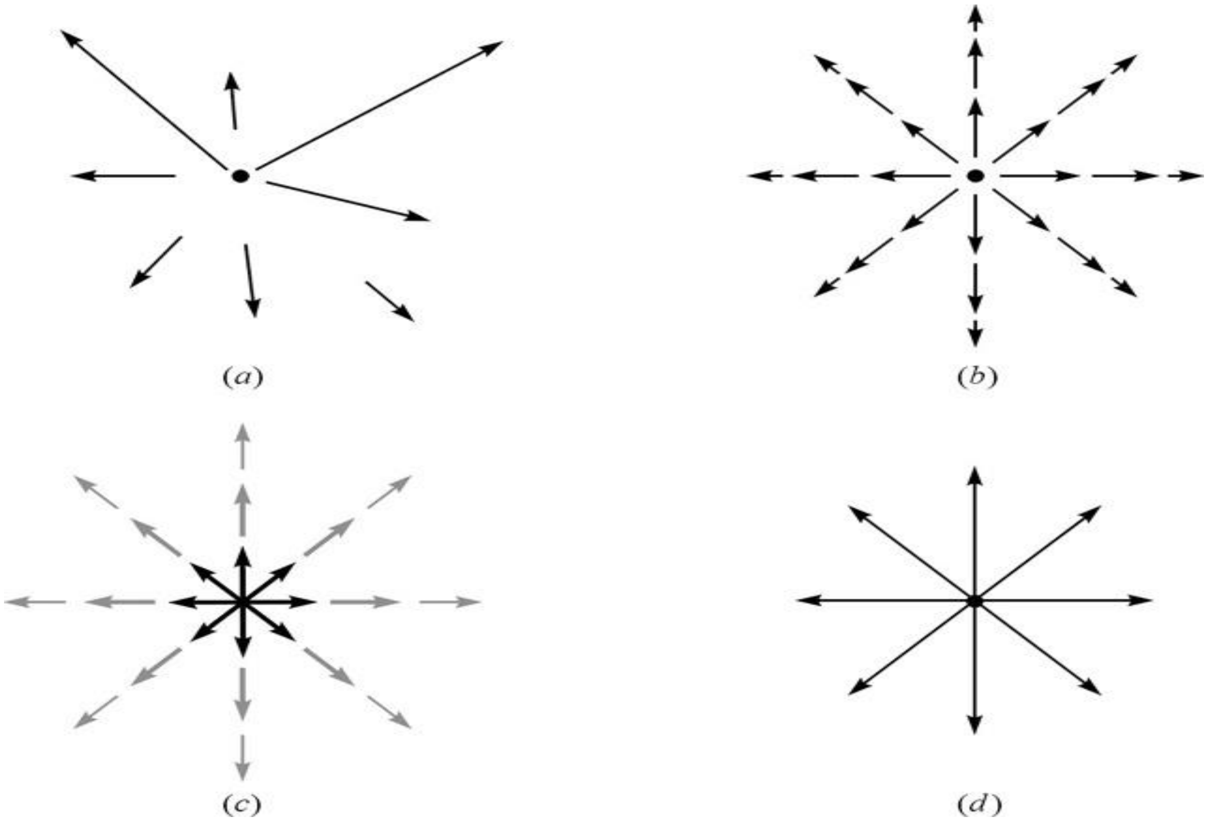
To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.

2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.

3. The conductor surface is an equipotential surface.

Streamlines and Sketches of Fields



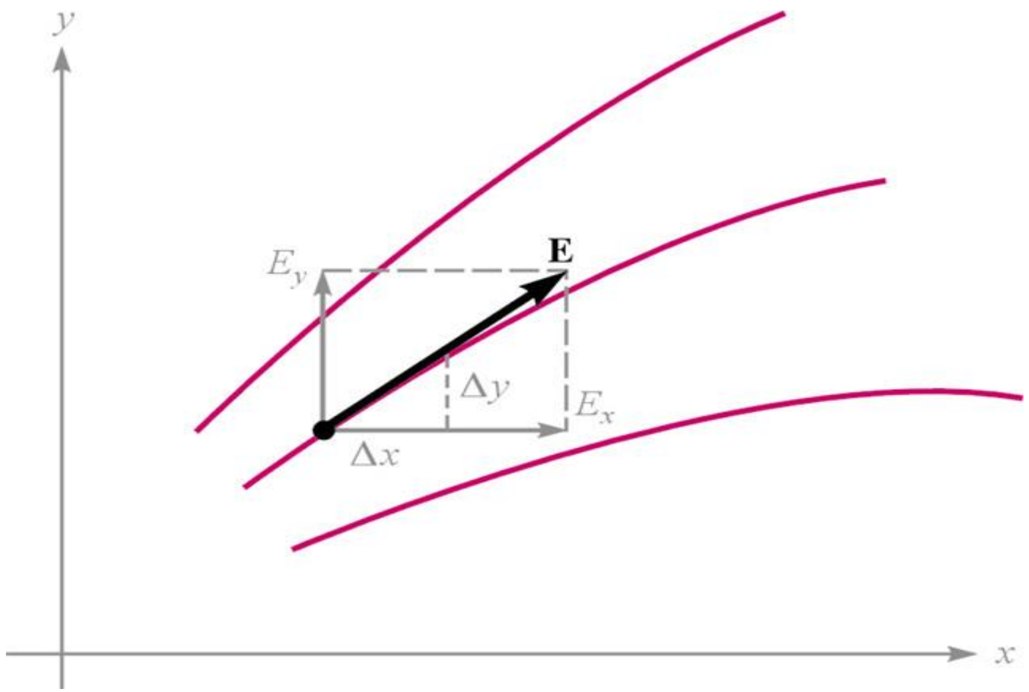
Cross-sectional view of the line charge.

Lengths proportional to the magnitudes of E and pointing in the direction of E

For the present, let us be content to show only the *direction* of **E** by drawing continuous lines, which are everywhere tangent to **E**, from the charge.

These lines are usually called streamlines, although other terms such as flux lines and direction lines are also used.

Streamlines and Sketches of Fields



If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially difficulties; for this reason sketching is usually limited to two-dimensional fields.

Several streamlines are shown in Figure, and the Ex and Ey components are indicated at a general point. It is apparent from the geometry that

$$\frac{E_y}{E_x} = \frac{dy}{dx}$$

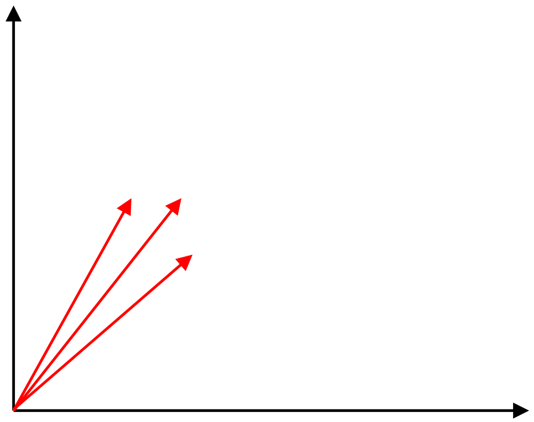
A knowledge of the functional form of Ex and Ey (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

Streamlines and Sketches of Fields

Consider the field about the line charge, $\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho \quad \Rightarrow \quad \mathbf{E} = \frac{x}{x^2+y^2}\mathbf{a}_x + \frac{y}{x^2+y^2}\mathbf{a}_y$$



Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \quad \text{or} \quad \frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + C_1 \quad \text{or} \quad \ln y = \ln x + \ln C$$

$$\boxed{y = Cx}$$

If we want to find the equation of one particular streamline, say one passing through P(-2, 7, 10), we merely substitute the coordinates of that point into our equation and evaluate C.

Streamlines and Sketches of Fields

Given the potential,

$$V = 100(x^2 - y^2)$$

and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-to-free-space boundary, find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S at P, and also the equation of the conductor surface.

Solution. The potential at point P is

$$V_P = 100[2^2 - (-1)^2] = 300 \text{ V}$$

Because the conductor is an equipotential surface, the potential at the entire surface must be 300 V. The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2) \qquad \text{or} \qquad x^2 - y^2 = 3$$

Streamlines and Sketches of Fields

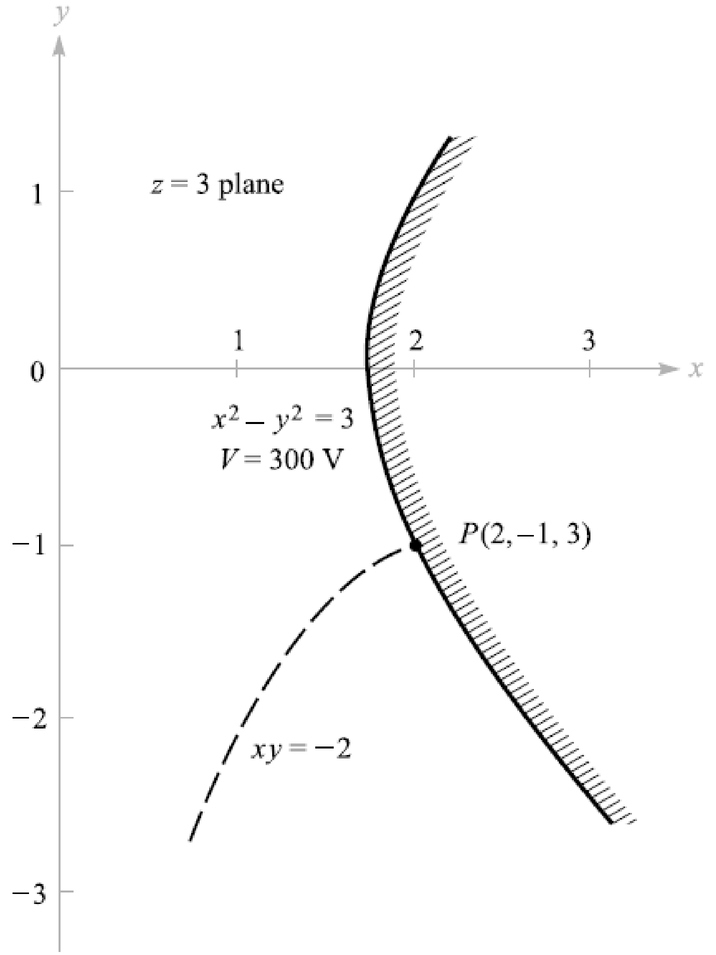
Next, we find E by the gradient operation,

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

At point P, $\quad \mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y$ V/m

Because $\mathbf{D} = \epsilon_0\mathbf{E}$, we have

$$\mathbf{D}_P = 8.854 \times 10^{-12}\mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \text{ nC/m}^2$$



The field is directed downward and to the left at P; it is normal to the equipotential surface. Therefore,

$$D_N = |\mathbf{D}_P| = 3.96 \text{ nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S,P} = D_N = 3.96 \text{ nC/m}^2$$

Streamlines and Sketches of Fields

Finally, let us determine the equation of the streamline passing through *P*.

$$\frac{E_y}{E_x} = \frac{200y}{-200x} = -\frac{y}{x} = \frac{dy}{dx}$$

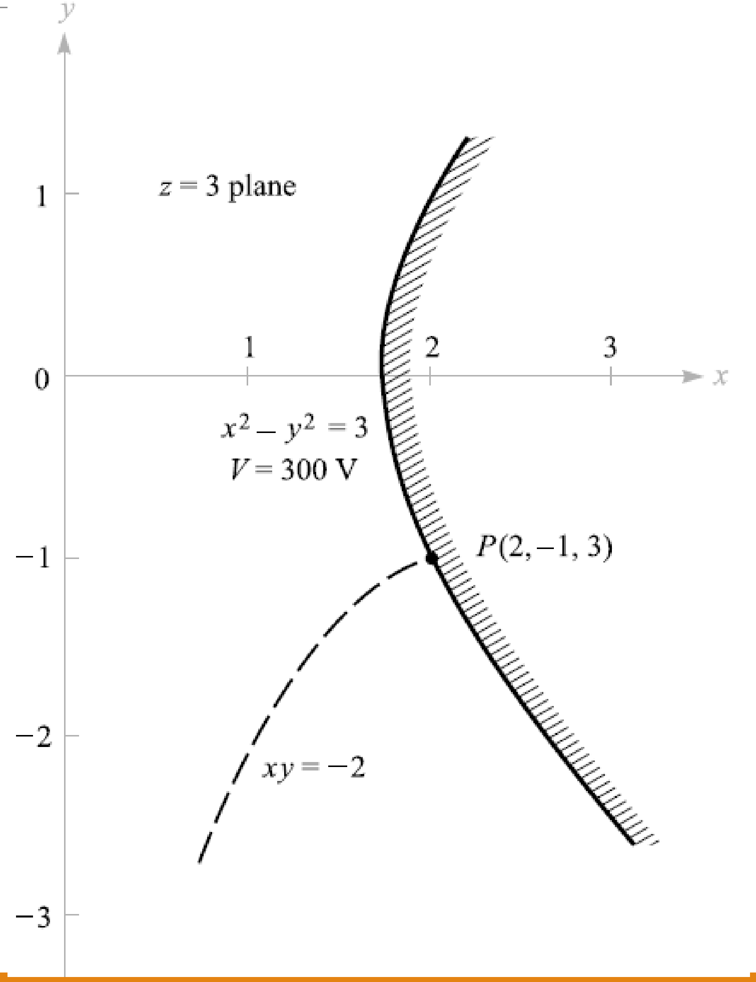
$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln y + \ln x = C_1$$

$$xy = C_2$$

The line (or surface) through *P* is obtained when $C_2 = (2)(-1) = -2$. Thus, the streamline is the trace of

$$xy = -2$$



The Electric Dipole

The dipole fields that we develop in this section are quite important because they form the basis for the behavior of dielectric materials in electric fields, as well as justifying the use of images theory.

An electric dipole, or simply a dipole, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to the distance to the point P at which we want to know the electric and potential fields.

The Electric Dipole

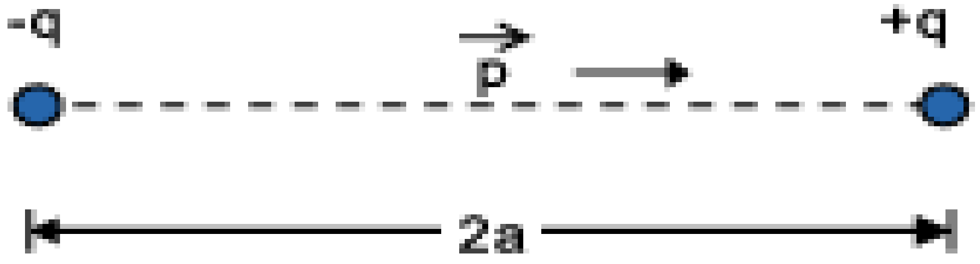
Dipole moment - a measure of the strength of electric dipole. It is a vector quantity represented by $\vec{P}$.

Magnitude of dipole moment - product of the magnitude of either charge and the separation between them.

Direction of dipole moment – it points from negative towards positive charge

$P = q\,(2a)$

SI unit – C m



The Electric Dipole

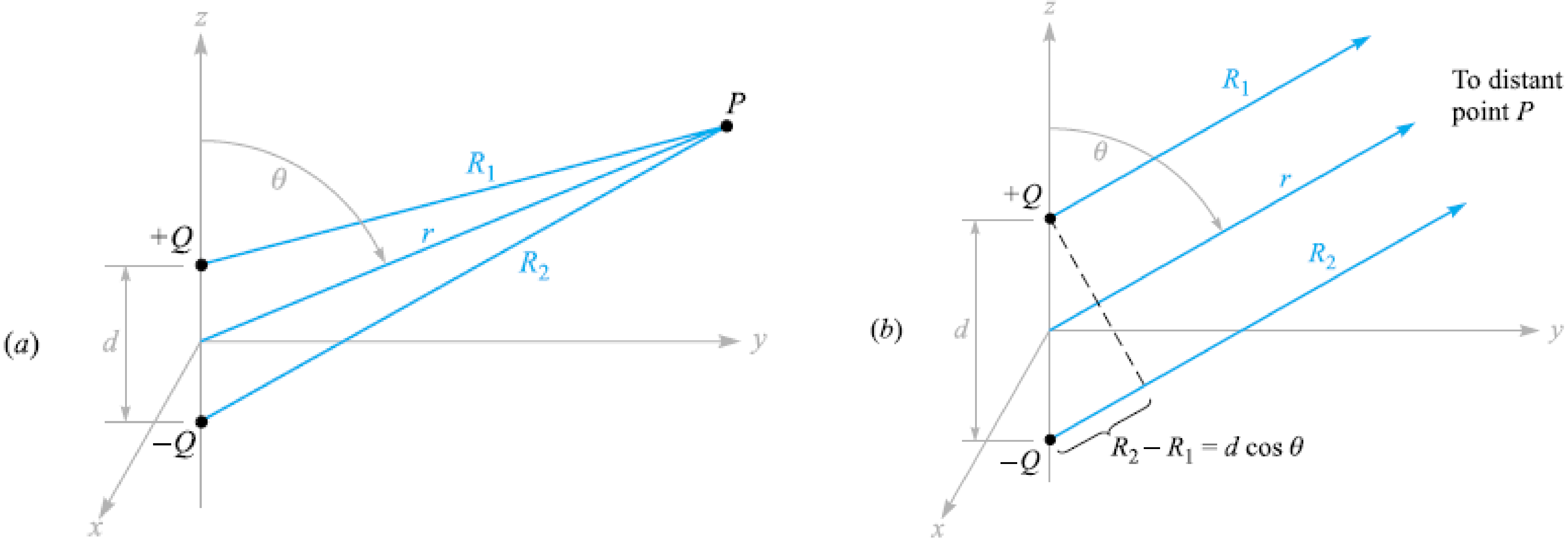


Figure 4.8 (a) The geometry of the problem of an electric dipole. The dipole moment p = Qd is in the a_z direction. (b) For a distant point P, R1 is essentially parallel to R2, and we find that R2 - R1 = d cosθ.

The Electric Dipole

Using the superposition to get the potential of point P due to the two charges

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

- For a distant point, $R_2 \approx R_1$, $R_2 - R_1 = d\cos\theta$
- The final result is then $$\boxed{V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}}$$
- Using the gradient relationship in spherical coordinates,

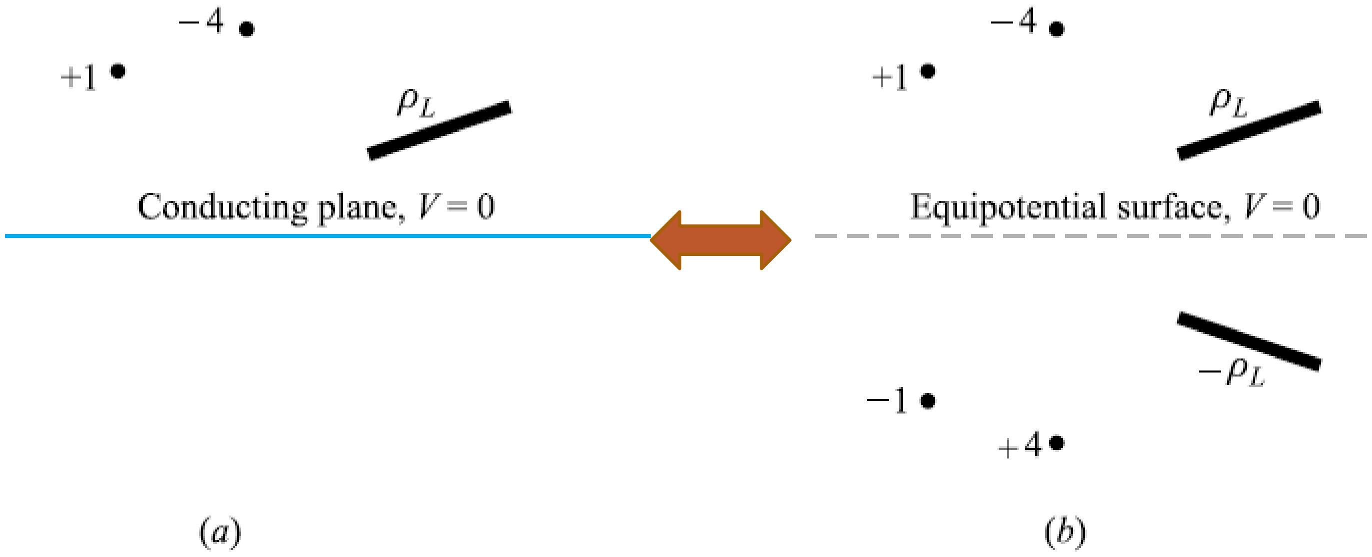
$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

$$\mathbf{E} = -\left(-\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\mathbf{a}_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\mathbf{a}_\theta\right) \Rightarrow \boxed{\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)}$$

The method of images

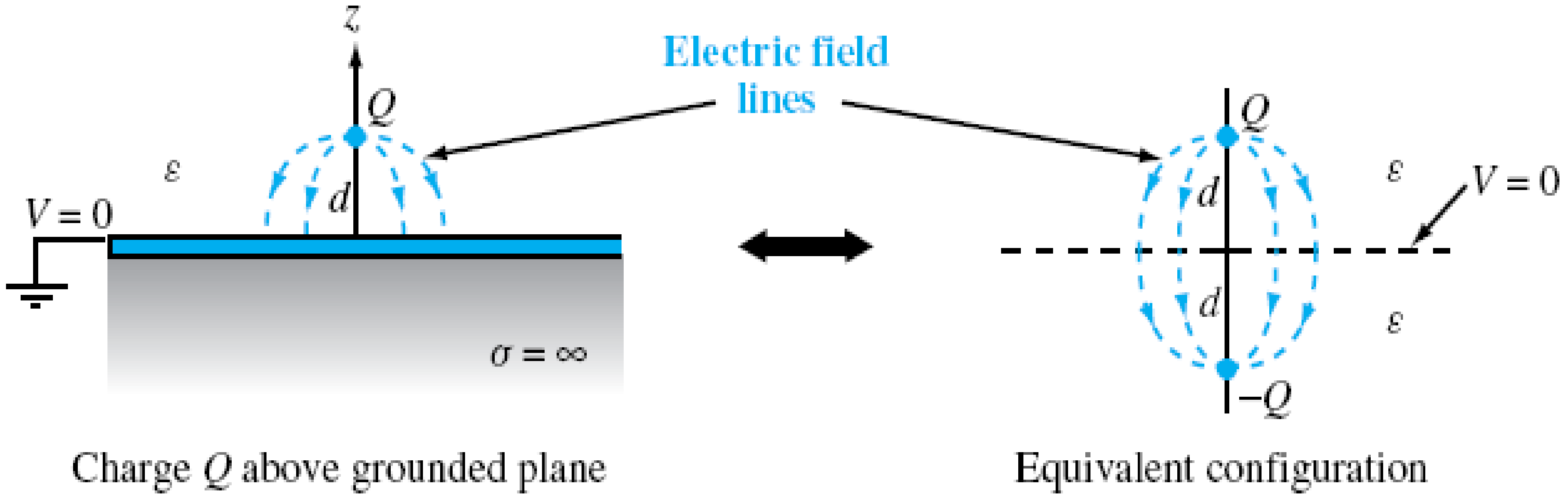
One important characteristic of the dipole field is the infinite plane at zero potential that exists midway between the two charges. Such a plane may be represented by a vanishingly thin conducting plane that is infinite in extent.

The conductor is an equipotential surface at a potential V = 0,



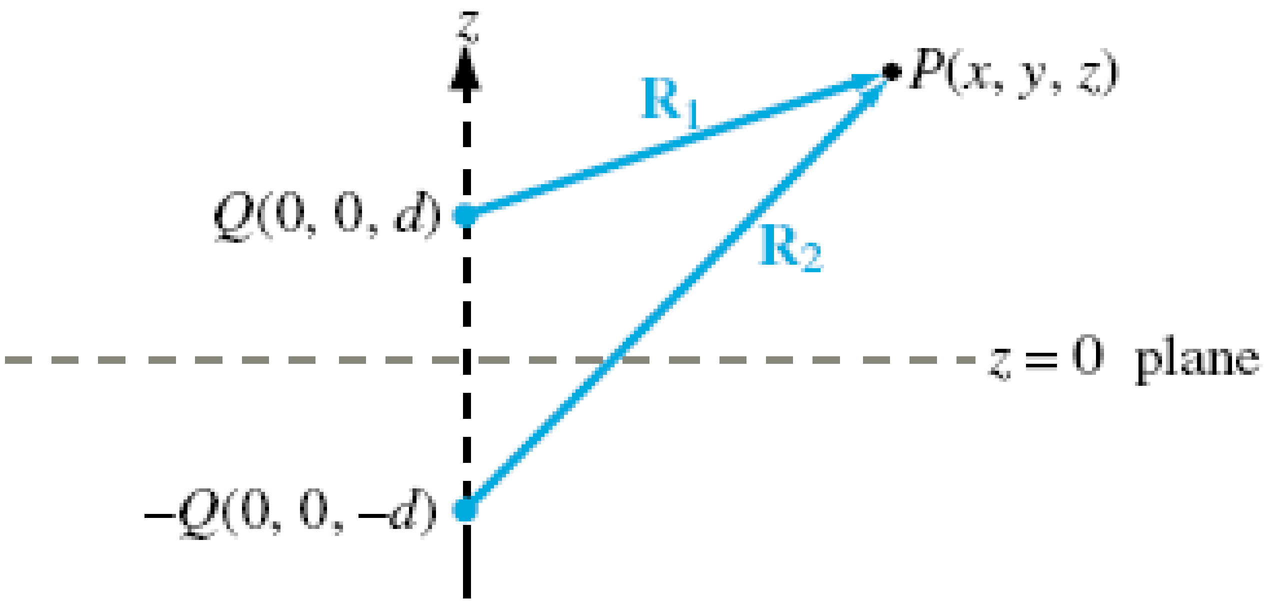
The method of images

Image theory states that a charge Q above a grounded perfectly conducting plane is equal to Q and its image $-Q$ with ground plane removed.



The method of images

Use image theory to determine **E** at an arbitrary point P (x, y, z) in the region $z > 0$ due to a charge Q in free space at a distance d above a **grounded conducting plane**.



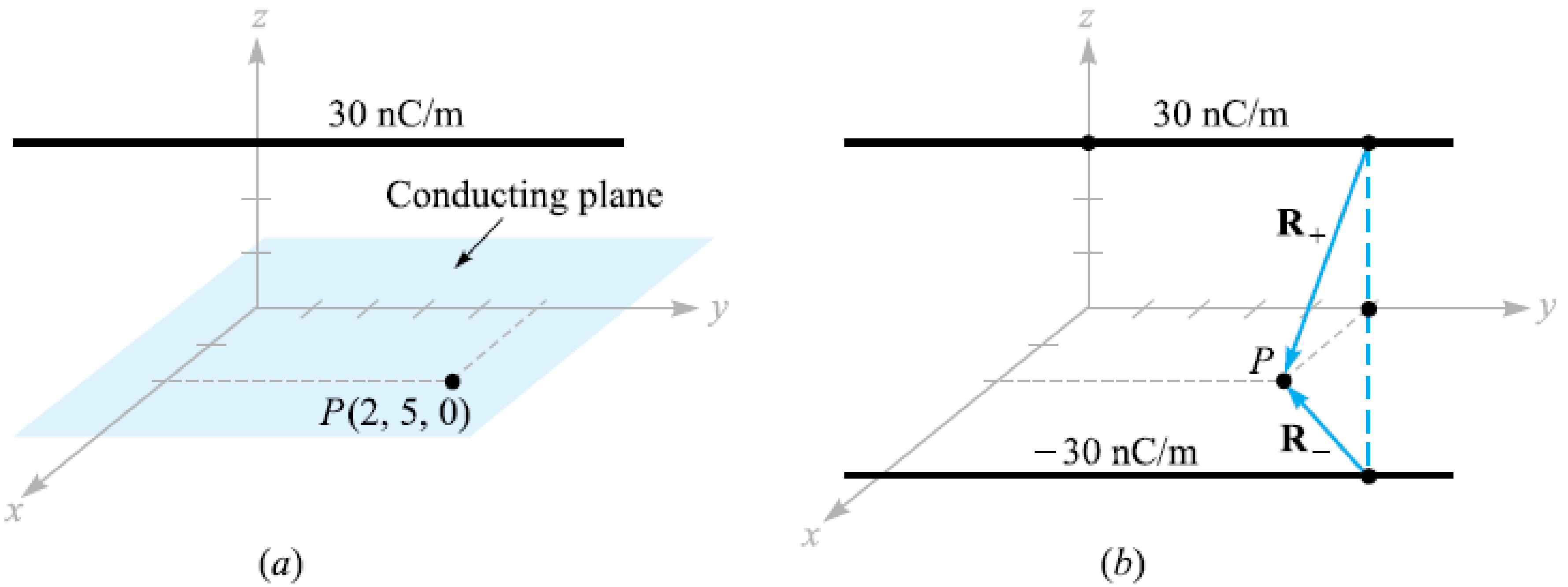
The method of images

Charge Q is at (0, 0, d) and its image $-Q$ is at (0,0,−d) in Cartesian coordinates. Using Coulomb's law, ***E*** at point P(x,y,z) due to two point charges:

$$E = \frac{1}{4\pi\varepsilon_0}\left(\frac{QR_1}{R_1^3} + \frac{-QR_2}{R_2^3}\right) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{x}x + \hat{y}y + \hat{z}(z-d)}{\left[x^2 + y^2 + (z-d)^2\right]^{3/2}} - \frac{\hat{x}x + \hat{y}y + \hat{z}(z+d)}{\left[x^2 + y^2 + (z+d)^2\right]^{3/2}}\right]$$

The method of images

find the surface charge density at P(2, 5, 0) on the conducting plane z = 0 if there is a line charge of 30 nC/m located at x = 0, z = 3, as shown in Figure



(a) A line charge above a conducting plane. (b) The conductor is removed, and the image of the line charge is added.

The method of images

We remove the plane and install an image line charge of -30 nC/m at x = 0, z = -3. The field at P may now be obtained by superposition of the known fields of the line charges.

The radial vector from the positive line charge to P is $R+ = 2a_x - 3a_z$, while $R- = 2a_x + 3a_z$.

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

$$\mathbf{E}_- = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have $$\mathbf{E} = \frac{-180\times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$

$D = \epsilon_0 E =$ $-2.20a_z$ nC/m^2, and because this is directed toward the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

The method of images

D5.6. A perfectly conducting plane is located in free space at $x = 4$, and a uniform infinite line charge of 40 nC/m lies along the line $x = 6$, $y = 3$. Let $V = 0$ at the conducting plane. At $P(7, -1, 5)$ find: (*a*) V; (*b*) $\mathbf{E}$.

Ans. 317 V; $-45.3\mathbf{a}_x - 99.2\mathbf{a}_y$ V/m